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(3 + *k*)-Dimensional Spacetime

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A mathematical model of the universe involving more than four dimensions is constructed. The $(3+k)$-spacetime is described, and the interaction theory of static charges is developed. An electromagnetic field theory is developed to show that Huyghen's principle and gauge invariance are not violated.

1. INTRODUCTION

Over 50 years ago Jeans (1935), during a discussion of the apparent indeterminacy of nature, told the following parable:

> Imagine, for instance, a race of blind worms whose perceptions were limited to the two-dimensional surface of the earth. Now and then spots of the earth would sporadically become wet. We, whose faculties range through three dimensions of space, call the phenomenon a rain-shower, and know that events in the third dimension of space determine, absolutely and uniquely, which spots shall become wet and which shall remain dry. But if the worms... tried to thrust all nature into their two-dimensional framework... (they) would only be able to discuss wetness and dryness of minute areas in terms of probabilities, which they would be tempted to treat as ultimate truth.

Jeans then went on to say that he felt that this illustrates the most promising interpretation of the situation in microphysics: a situation which still exists today.

The present paper is the outcome of an attempt to construct a mathematical model of the universe involving more than four dimensions that agrees with macroscopic theory and can be used as a vehicle to test whether a situation as visualized by Jeans can exist.

It turns out that postulating more dimensions is not as straightforward as was first envisaged, due to two principles, that of Huyghens and that of gauge invariance. In order to avoid such difficulties, it was found sufficient

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to assume that spacetime comprises a union of subspaces of four dimensions, which we call L-spaces. The contents of the universe are supposed to be scattered among these subspaces; however, a given observer, through whose senses and equipment the universe is described, will have perceptions limited to one only of these subspaces. This observer will, accordingly, be in much the same situation as a worm-scientist of Jeans' parable.

Section 2 is devoted to a description of what we have called $(3+k)$-spacetime, and, as an application, the interaction theory of static charges is developed in Section 3. It is here that Jeans' prediction appears to be true. This section includes an attempt to determine k.

In order to show that Huyghen's principle and the principle of gauge invariance are not violated, it is necessary to develop an electromagnetic field theory to some extent, and this is done in Section 4.

2. DESCRIPTION OF $(3+k)$-SPACETIME

Weyl (1949) pointed out that "intuitively evident meaning can only be attached to spatio-temporal proximity and coincidence." Accordingly, the world-points under consideration will be assumed to be neighboring in order to assist in this initial description. This enables an observer to represent the usual 4-dimensional spacetime [or $(3+1)$-spacetime] locally by a diagram such as is shown in Figure 1. The lines $t=$ const are more commonly replaced by planes in such a diagram, but we need a perspective diagram for our developments, so in Figure 1 the three spatial dimensions have been crowded into a single dimension. The dashed line $s=$ const represents a point fixed in space according to the observer at different epochs t. A body permanently located at this point would have a world-line coincident with the $s=$ const line. Such a line is called a *fiber*. Clearly there is a continuum infinity of such fibers.

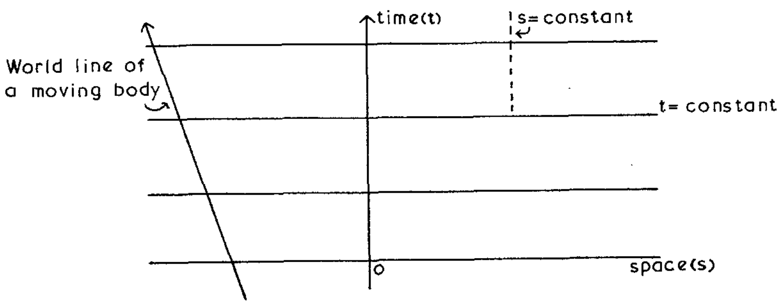


Fig. 1. Representation of a small region of $(3+1)$-spacetime.

Weyl describes the situation very succinctly: (a) All simultaneous world-points ($t = \text{const}$) form a 3-dimensional stratum, (b) all world-points of equal location ($s = \text{const}$) form a 1-dimensional fiber.

In what follows the word *location* will always refer to a place in ordinary 3-dimensional space.

The possibilities for carrying out Jeans' suggestion of increasing the dimensionality of spacetime are severely restricted by the following two results: the first is that the wave equation (in n spatial dimensions) shows that Huyghen's principle can only hold when n is odd (Courant and Hilbert, 1966). Put more picturesquely, one can only be sure that darkness follows when the candle is extinguished if the number of spatial dimensions is odd (Weyl, 1949). The second result is that the principle of gauge invariance holds if, and only if, spacetime is 4-dimensional (Weyl, 1949).

On combining these two results with the fact that common experience shows that there are at least three spatial dimensions, one sees that if the two principles are to be preserved, then the spacetime of the observer must indeed be a $(3+1)$-spacetime.

At first sight this seems to rule out Jeans' suggestion. However, we shall proceed by modifying Weyl's description of spacetime by leaving (a) unchanged and replacing (b) by the following:

(b′) All world-points of equal location form a k-dimensional fiber space ($k > 1$).

It will turn out that the two principles mentioned above can now be preserved (Section 4) and at the same time a certain type of apparently spontaneous event no longer appears to be spontaneous, as was forecast by Jeans (Section 3).

In the new description of spacetime, namely (a) and (b′), the term *fiber space* implies that through every point of the space there passes at least one fiber. Our fiber space is a normed vector space, and in what follows in Section 3 the norm is assumed to be locally Euclidean.

To each fiber there corresponds a $(3+1)$-spacetime which is locally Lorentzian. We call these spacetimes Lorentz spacetimes or, more briefly, L-spaces. The L-spaces are embedded in our $(3+k)$-speactime; this latter we denote by σ. Note that all the L-spaces of σ have a common 3-space.

The diagram of Figure 1 is now replaced by that of Figure 2. In Figure 2 only two L-spaces are shown, L_1 and L_2, with their respective fibers [1] and [2] at the particular location $s = \text{const}$, which is represented in the diagram by an infinite plane.

A body is said to *belong* to (or to be *in*) a given L-space if a fiber of the L-space passes through each point of its world-line. In particular an observer considers itself at rest at the location $s = \text{const}$ in a given L-space, L_1 say, if its world-line coincides with the fiber [1]. Furthermore, the same

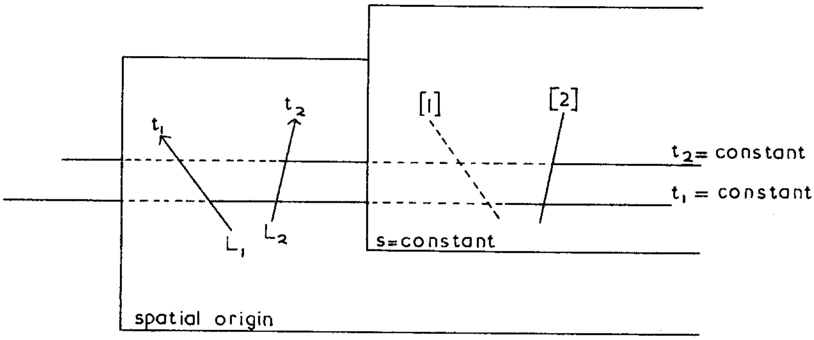


Fig. 2. A representation of a small region of σ.

observer can mark off a time scale along that fiber of L_1 that is located at the chosen spatial origin. This scale is shown as t_1 in Figure 2. The scale t_2 has been determined by an observer belonging to L_2. The epochs $t_1 = 0$, $t_2 = 0$ are chosen arbitrarily by the respective observers.

3.1. INTERACTION THEORY

3.1. Electrostatic Example

In order to construct a theory from the hypotheses (a) and (b′) of the last section, a number of assumptions will be made. These will be labeled $A_1, A_2, ..$ and will be introduced at the appropriate stages in the development of the theory.

A1. An observer belonging to a given L-space, when making observations of events associated with bodies[2] belonging to other L-spaces, will interpret the results as if his is the only L-space that exists.

The statistical nature of present-day physical theory is a direct outcome of this assumption if the universe does in fact consist of interacting subspaces.

Let us now suppose that a point charge of strength q coulombs belonging to the L-space L_1, say, produces an electric flux at world-points of the observer's L-space, which we denote by L_2.

Consider three world-points A in L_1 and B and C in L_2. Suppose that the spatial parts of A and B are located at the origin and let C have spatial coordinates (x_1, x_2, x_3). For simplicity, B and C are supposed mutually at rest according to the observer in L_2.

A world-point in σ has coordinates $(x_1, x_2, x_3, \theta_1, \theta_2, \cdots, \theta_k)$, which are abbreviated to $(\mathbf{x}, \boldsymbol{\theta})$, where the bold letters refer either to 3- or k-vectors.

[2]We use the term *body* in the general sense of any entity with a world-line.

The $(3+k)$-dimensional *world vectors* appearing in the next section will be denoted by script letters.

The coordinate (x_1, x_2, x_3) are spatial; that is, they refer to a point in ordinary space. The coordinates $(\theta_1, \theta_2, \ldots, \theta_k)$ are temporal, and refer to a point in fiber space.

It is shown in the Appendix that given any two (straight) fibers from different L-spaces but at the same location, then to every point of one fiber there corresponds a point on the other that is to be regarded as *simultaneous.*

Let the observer choose a time scale along the fiber [2] (Figure 3) such that at 0, $t=0$. The event B occurs at a time interval t seconds later. Also, let the k-vectors $\boldsymbol{\theta}$ have 0 as origin.

Figure 4 shows the fibers [1] and [2] in fiber space; this is more fully illustrated in Figure 8.

Let M on [2] and M' on [1] be the simultaneous pair that are nearest (see Appendix) to each other in fiber space and denote $\overline{MM'}$ by a. It is convenient to define unit k-vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ by the formulas (see Appendix)

$$\boldsymbol{\theta}_0 = \mathbf{0}, \qquad \boldsymbol{\theta}_M = t_0\mathbf{v}, \qquad \boldsymbol{\theta}_B = t\mathbf{v}$$

$\boldsymbol{\theta}_{M'} - \boldsymbol{\theta}_M = a\mathbf{w}$ (since M and M' are simultaneous, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular) and

$$\boldsymbol{\theta}_A - \boldsymbol{\theta}_{M'} = (t - t_0)\mathbf{u}$$

It follows that

$$\begin{aligned}\overline{AB}^2 = |\boldsymbol{\theta}_A - \boldsymbol{\theta}_B|^2 &= |(\boldsymbol{\theta}_A - \boldsymbol{\theta}_{M'}) + (\boldsymbol{\theta}_{M'} - \boldsymbol{\theta}_M) + \boldsymbol{\theta}_M - \boldsymbol{\theta}_B|^2 \\ &= |(t - t_0)(\mathbf{u} - \mathbf{v}) + a\mathbf{w}|^2 \\ &= (t - t_0)^2\alpha^2 + 2a(t - t_0)\beta + a^2\end{aligned}$$

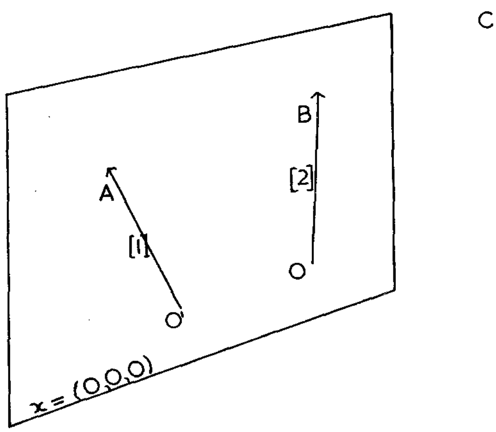


Fig. 3. The simultaneous trio A, B, C and the simultaneous pair 0, 0′ in the spacetime σ.

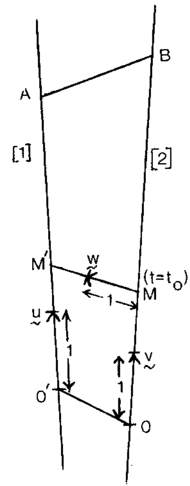


Fig. 4. The unit k-vector $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in fiber space.

We call $\alpha = |\mathbf{u} - \mathbf{v}| = [2(1 - \mathbf{u} \cdot \mathbf{v})]^{1/2}$ and $\beta = \mathbf{u} \cdot \mathbf{w}$ the *obliqueness factors.* Note that since $\mathbf{v} \cdot \mathbf{w} = 0$, then if $\mathbf{u} \cdot \mathbf{v} = 1$, so that $\mathbf{u} = \mathbf{v}$, we have

$$\alpha = \beta = 0$$

In this case the L-spaces L_1 and L_2 are said to be parallel. If, in addition, $a = 0$, L_1 and L_2 are identical.

The world-point C, which belongs to the same L-space as B (Figure 3), is defined to be simultaneous with B in the normal manner (see, e.g., Eddington, 1946). This means that $\boldsymbol{\theta}_C = \boldsymbol{\theta}_B$. We shall also denote the spatial separation between C and A by r. It follows that the interval between A and C is given by

$$\overline{AC}^2 = |\mathbf{x}|^2 - c^2|\boldsymbol{\theta}_C - \boldsymbol{\theta}_A|^2 = r^2 - c^2\overline{AB}^2$$

In obtaining this formula it was noted that $\boldsymbol{\theta}$ has the dimensions of time, and in order to made the spacetime σ compatible with, say, that of Minkowski, $\boldsymbol{\theta}$ has been multiplied by the usual factor ic, where c is the speed of light through ordinary space.

If we now write

$$s = c[(t - t_0)^2\alpha^2 + 2a(t - t_0)\beta + a^2]^{1/2}$$

$$\boldsymbol{\xi} = c(\boldsymbol{\theta}_C - \boldsymbol{\theta}_A) = c(\boldsymbol{\theta}_B - \boldsymbol{\theta}_A)$$

then C has the real coordinates $(\mathbf{x}, \boldsymbol{\xi})$ with A as the origin $(\mathbf{0}, \mathbf{0})$ in σ. Also, C lies on a member of the family of hyperhyperboloids

$$x_1^2 + x_2^2 + x_3^2 - \xi_1^2 - \xi_2^2 - \cdots - \xi_k^2 = r^2 - s^2 = \overline{AC}^2$$

Note, in particular, that the interval $\overline{AC}$ is fixed over any member of the given hyperhyperboloid family in σ.

Suppose, now, that at a certain epoch a point charge q is at A in L_1. In order to calculate the magnitude of its flux density at C in L_2, the usual concept of flux distribution will have to be extended, as mentioned earlier. We introduce the following two assumptions, which are extensions of the usual ones for an isolated charge at the origin:

A2. The magnitude of the flux density at a world-point C simultaneous with the world-point A is a function of the interval $|\overline{AC}|$ only; the sign of the flux is that of the charge q.

A3. The flux lines are normal to the family of concentric spheres in ordinary space, $|\mathbf{x}| = \text{const}$, and are such that $\boldsymbol{\theta} = \text{const}$ along their lengths. This condition implies that $\boldsymbol{\xi} = \text{const}$ along a flux line.

As C roves over a given member of the family of hyperhyperboloids, the magnitude of the flux density at C will remain constant by A2.

Condition A3 appears less arbitrary if the usual 1-dimensional fiber space is considered, for in this case the 1-vector $\boldsymbol{\theta}$ becomes $t\mathbf{u}$, where $\mathbf{u}$ is a fixed unit vector. The condition $\boldsymbol{\theta} = \text{const}$ becomes the same as the statement that t is constant along flux lines in electrostatics: that is, flux lines do not extend into the past or future.

Figure (5) shows a plot of $|\boldsymbol{\xi}|$ versus $|\mathbf{x}|$ for three members of the corresponding family of hyperbolas. In particular, the asymptote $|\mathbf{x}|^2 - |\boldsymbol{\xi}|^2 = 0$ corresponds to the asymptotic hypersurface $x_1^2 + x_2^2 + x_3^2 - \xi_1^2 - \xi_2^2 - \cdots \xi_k^2 = 0$. This passes through the origin A $(\mathbf{0}, \mathbf{0})$ in σ occupied by the point charge, and where the magnitude of the flux density is therefore infinite. It follows from A2 that there is infinite flux density at every point of the asymptotic

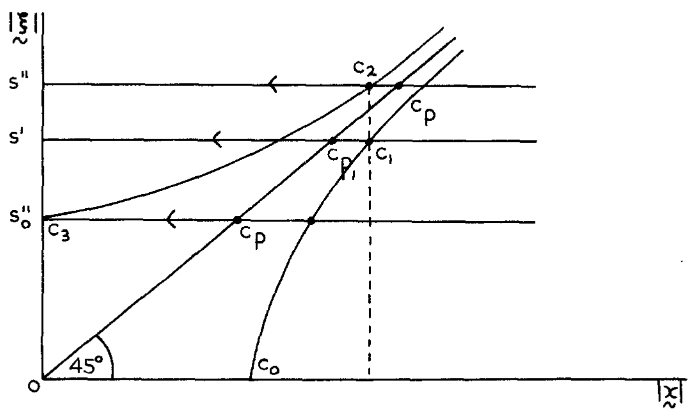


Fig. 5. Graph of $|\mathbf{x}|^2 - |\boldsymbol{\xi}|^2 = r_C^2 - s_C^2$ for various positions of C in σ, with $q > 0$.

hypersurface. Of course, if C lies on this hypersurface, the interval $\overline{AC}$ is zero.

Again A3 shows that the flux is represented in Figure 5 by arrowed horizontal lines, only three of which are shown.

Suppose now that C has coordinates $\mathbf{x}_c$, $\boldsymbol{\xi}_c$ such that $|\mathbf{x}_c| = r'$, $|\boldsymbol{\xi}_c| = s'$. The corresponding point is marked C_1. The flux will appear to the observer to emanate from a charge of magnitude p situated not at the spatial point occupied by q, but at a point distant $r' - s'$ away from the observer, represented by C_{p_1} in Figure 5. Since Figure 5 only gives s' as the spatial coordinate of p, it might be argued that p could be distributed over a sphere of radius s'. However, because C_{p_1} lies on the asymptote, the flux density, and hence the apparent charge density, is infinite. Finte total flux requires p to be a point charge.

Let the magnitude of the apparent charge seen from $C_1(r', s')$ be

$$p_1 = \lambda_1 q$$

where λ_1 is to be determined, and call this the *apparent source* of the flux.

If D is the magnitude of the flux density at C, then, by A3, integration over the surface of the sphere of radius $\overline{C_{p_1}C_1} = r' - s'$ gives the charge

$$p_1 = 4\pi D(r' - s')^2$$

On the other hand, the apparent source is at A, coinciding with the true source (so that $p = q$), when C lies at a point corresponding with C_0 in the diagram. Integration over the sphere of radius $\overline{0C_0}$ gives

$$q = 4\pi D(r'^2 - s'^2)$$

Assumption A_2 ensures that D is the same in both formulas. On combining the two formulas we get

$$\lambda_1 = \frac{r' - s'}{r' + s'} \qquad (r' > s')$$

If r' is fixed, the passage of time will vary s away from s'. Thus, if we allow s to increase, λ will decrease to zero. The question now arises: What happens to λ as s continues to increase?

Suppose s reaches a value s''. Now C will have coordinates such that $|\mathbf{x}_C| = r'$, $|\boldsymbol{\xi}_C| = s''$, with $r' < s''$. The corresponding point in Figure 5 is C_2, and it lies on a branch of the family of hyperbolas with imaginary $\overline{AC}$. Let this cut the $|\boldsymbol{\xi}|$ axis at s''_0. Since this hyperbola is $|\mathbf{x}|^2 - |\boldsymbol{\xi}|^2 = r'^2 - s''^2$, then s''_0 is obtained by substituting $|\mathbf{x}| = 0$. Hence $s''^2_0 = s''^2 - r'^2$.

Hypothesis A_2 ensures that the flux density at C_2 is the same as that at C_1 provided that $s''_0 = \overline{0C_0}$.

If D is the magnitude of the flux density at these points, then the apparent charge seen from a point on the surface represented by $C_3(0, s_0'')$ is

$$p_3 = 4\pi D s_0''^2 = 4\pi D \cdot \overline{0C}_0^2 = q$$

The apparent charge seen from a point on the surface represented by $C_2(r', s'')$ is

$$p_2 = 4\pi D(s'' - r')^2 = (g/s_0''^2)(s'' - r')^2$$

But $s_0''^2 = s''^2 - r'^2$, and so

$$p_2 = \frac{s'' - r'}{s'' + r'} q = \lambda_2 q, \quad \text{say}$$

Comparing this with the formula for λ, we see that in general

$$\lambda = \frac{|r - s|}{r + s}$$

3.2. Determination of k

We begin by looking at our model from a slightly different point of view (Figure 6). Although we have used the same labels in Figure 6 as in Figure 5, the hyperbolas shown in the two figures need not be the same. In

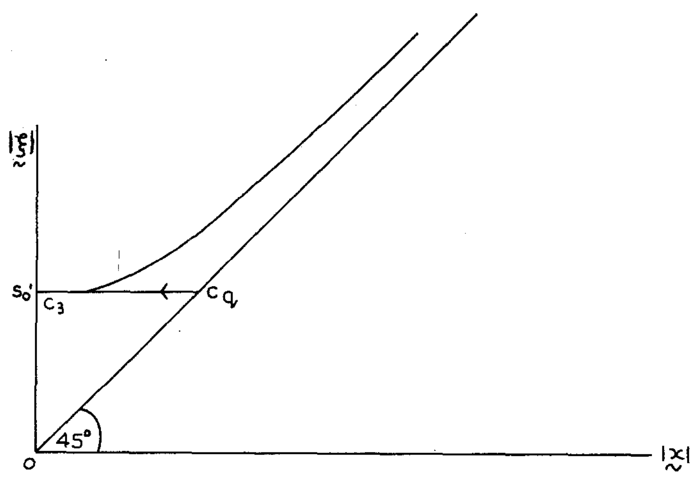


Fig. 6. The point C_1 representing the charge q as seen from a point on the surface represented by $C_3(0, s_0'')$.

the case shown in Figure 6 all the flux lines from q touch the hyperhyperboloid at its intersection with the k-flat $x_1=0$, $x_2=0$, $x_3=0$, namely the hypersphere

$$\xi_1^2+\xi_2^2+\cdots+\xi_k^2=-\overline{AC}^2=s_0''^2\geq 0$$

This is represented by C_3 in the figure.

Remembering that $|\overline{AC}|$ is now the radius of this hypersphere, we see that assumption A2 implies that the flux lines from q meet it in points that are distributed uniformly over its surface with a density of magnitude D', say.

We can now obtain q by integrating D' over the surface of the hypersphere. The result is

$$q=\frac{2\pi^{k/2}}{\Gamma(k/2)}D'(s''^2-r'^2)^{(k-1)/2}$$

We return to the method used in Section 3.1; since q is apparently situated at the center of a sphere in ordinary space of radius $\overline{C_3C_q}$ (see Figure 6) with C on the hyperhyperboloid, we see that the density D' at C is the same as in the above formula, and also that q can be obtained, using the assumption A3, by integrating D' over this sphere. The result is

$$q=4\pi D'(s''^2-r'^2)$$

These two formulas are consistent if, and only if, $k=3$.

We conclude that σ is a 6-dimensional (3+3)-spacetime.

Strictly, this result merely states that fiber space and ordinary space have the same number of dimensions. If some future investigation suggests that the number of spatial dimensions should be changed, then a similar change must be made in the number of dimensions of fiber space.

A rather striking consequence of the formulas so far developed is illustrated by the equation

$$q=4\pi D|r^2-s^2|$$

where the primes have been dropped. The parameter s is time-dependent, but the shape of the s versus t curve depends upon the relative magnitudes of t_0, a, and the obliqueness factors. Two examples are shown in Figure 7a.

If L and L_2 are parallel, then $s=ca$, and there is no time variation.

For nonparallel L-spaces we expect a time variation in D, since q is fixed, by hypothesis. The behavior of D with time depends upon the initial values, and Figure 7b illustrates three examples.

The singularities at $t=t_1$ and r_2, for example, are represented by the asymptote of Figure 5.

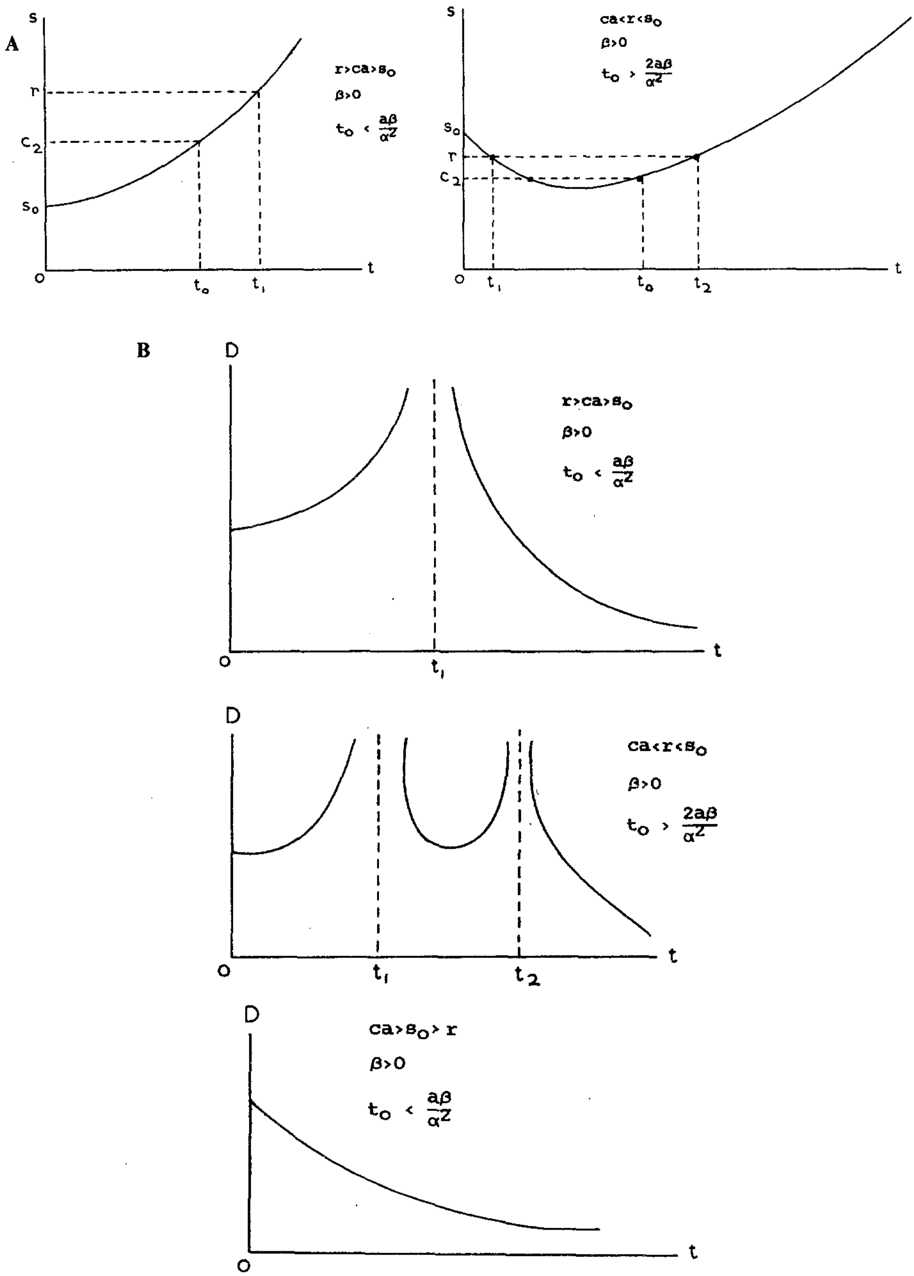


Fig. 7. (a) The difference in shape of the s versus t curve in two particular cases. (b) The variation of D with time when r is fixed in three different cases.

If the L-spaces are parallel, t_1 moves off to $-\infty$ and t_2 to $+\infty$ and the flux density is such that

$$D = \frac{q}{4\pi(r^2 - c^2a^2)}$$

showing no time variation.

Now suppose that the observed flux density is accompanied by an observed electric field of magnitude

$$E = D/\varepsilon_0$$

where ε_0 is the permittivity of the observer's L-space. If there is no polarizable medium present, the electric field $\mathbf{E}$ and the flux density $\mathbf{D}$ are parallel and have the same direction as the radius vector to the point of observation in ordinary space from the origin $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, which is the location of the charge q. Accordingly $\mathbf{D} = D\mathbf{r}/r$ and so the electric field is

$$\mathbf{E} = \frac{q\mathbf{r}}{4\pi\varepsilon_0 r(r^2 - s^2)}$$

It is readily verified that the scalar potential function

$$\phi = \frac{q}{8\pi\varepsilon_0 s}\log\frac{r+s}{|r-s|}$$

satisfies the usual electrostatic formula

$$\mathbf{E} = -\nabla\phi$$

It follows at once that the principle of superposition holds.

Consider now a *charge distribution* in L_1 contained within a fixed spatial volume V. If the charge density is ρ_1 within an element of volume dV then the potential at C in L_2 due to the charge in dV belonging to L_1 is

$$d\phi = \frac{\rho_1\, dV}{4\pi\varepsilon_0 R}$$

where

$$R = R(r, s) = \frac{2s}{\log[(r+s)/|r-s|]}$$

Note that $R(r, 0) = r$.

From the superposition principle the potential at C due to the whole distribution in L_1 is

$$\phi = \frac{1}{4\pi\varepsilon_0}\iiint\limits_V \rho_1 \frac{dV}{R}$$

where in order to carry out the integration the spatial origin has been transferred from the elementary charge to the observed point C in L_2.

This potential does not satisfy the Poisson equation $\nabla^2\phi = -\rho_1/\varepsilon_0$.

In view of the time dependence of R, there is a secular change in ϕ (and hence in the corresponding **E**) at C, even when ρ_1 is fixed. However, as we saw earlier, the observer will not assign a charge density of ρ_1 at a given point. In fact, the observer will replace the elementary charge $\rho_1\, dV$ by an apparent charge $\lambda\rho_1\, dV$ in a volume element dV', say, located at a different site. Thus, if ρ_{12} is the apparent charge density in L_2 due to a charge distribution in L_1, then $\rho_{12}\, dV' = \lambda\rho_1\, dV$. If r is the distance between $\rho_1\, dV$ and C in ordinary space, then the corresponding distance between $\rho_{12}\, dV'$ and C is, as we have seen earlier (vide Figure 5), $r' = |r - s|$.[3] Writing

$$\nabla' = \left\{\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}, \frac{\partial}{\partial x_3'}\right\}$$

where (x_1', x_2', x_3') are the spatial coordinates of $\rho_{12}\, dV'$ with $dV' = dx_1'\, dx_2'\, dx_3'$ and C as origin, we have at once that $\nabla'^2(1/r') \equiv 0$, and so from the usual argument from one of Green's theorems the formula

$$\phi = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_{12}}{r'}\, dV'$$

is a solution to Poisson's equation $\nabla'^2\phi = -\rho_{12}/\varepsilon_0$.

Our observer (who is in much the same situation at Jeans' worm-scientist) will, from his much more common experience of charge distributions belonging to his own L-space (when $R = r = r'$), assume that all detectable fields and charge distributions satisfy Poisson's equation, thereby thrusting the apparent charge density ρ_{12} into the above framework in accordance with A1. Accordingly, he will use the formula

$$\phi = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_{12}}{r'}\, dV'$$

rather than

$$\phi = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_1}{R}\, dV$$

There is a secular change in ρ_{12} because of the factor λ.

[3]Here r' is different from that in Figure 5.

Finally, if there is also present a charge distribution of density ρ_2 in L_2 occupying the same fixed volume V, then by superposition the potential ϕ is given by the observer as

$$\phi = \frac{1}{4\pi\varepsilon_0}\iiint\limits_V \frac{\rho_{12}+\rho_2}{r'}\,dV'$$

with $\nabla'^2\phi = -(\rho_{12}+\rho_2)/\varepsilon_0$.

Of course the observer sees no secular change in ρ_2.

4. ELECTROMAGNETIC THEORY

The fact that our observer is describing his results in terms of his own L-space, whichis a $(3+1)$-spacetime, suggests strongly that Huyghen's principle and the principle of gauge invariance hold as far as he is concerned. To check that our theory agrees with this, we develop electromagnetic theory in σ a short way. We find that we shall need one further assumption, namely A4, which will be given shortly.

In the development of the field equations it is convenient to use a form of vector product employed by Rutherford (1943). If $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B}(b_1, b_2, b_3)$ are any two 3-vectors, and

$$C_{12} = a_1 b_2 - a_2 b_1, \qquad C_{23} = a_2 b_3 - a_3 b_2, \qquad C_{31} = a_3 b_1 - a_1 b_3$$

then the usual definition of vector product is

$$\mathbf{A}\times\mathbf{B} = (C_{23}, C_{31}, C_{12})$$

On the other hand, Rutherford's definition, which we call the *matrix vector product*, is defined by the skew symmetric matrix

$$\mathbf{A}\wedge\mathbf{B} = \begin{pmatrix} 0 & C_{12} & C_{13} \\ C_{21} & 0 & C_{23} \\ C_{31} & C_{32} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \mathbf{\Gamma}_3 \end{pmatrix}$$

where $\mathbf{\Gamma}_1 = (0, C_{12}, C_{13})$, $\mathbf{\Gamma}_2(C_{21}, 0, C_{23})$, $\mathbf{\Gamma}_3 = (C_{31}, C_{32}, 0)$.

The dot product between two vectors is the usual scalar product

$$\mathbf{A}\cdot\mathbf{D} = a_1 d_1 + a_2 d_2 + a_3 d_3$$

Thus

$$\mathbf{D}\cdot(\mathbf{A}\times\mathbf{B}) = d_1 C_{23} + d_2 C_{31} + d_3 C_{12}$$

is a scalar, whereas

$$\mathbf{D}\cdot(\mathbf{A}\wedge\mathbf{B}) = (\mathbf{D}\cdot\mathbf{\Gamma}_1, \mathbf{D}\cdot\mathbf{\Gamma}_2, \mathbf{D}\cdot\mathbf{\Gamma}_3)$$

is a vector. If **A** and **D** are both replaced by the operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, then $\nabla \cdot (\nabla \times \mathbf{B}) \equiv 0$ and

$$\nabla \cdot (\nabla \wedge \mathbf{B}) \equiv \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \equiv \nabla \times (\nabla \times \mathbf{B})$$

We call $\nabla \wedge \mathbf{B}$ the matrix curl of **B**.

Rutherford used the matrix definition in his treatment of 4-vectors, and we shall follow his lead in the development of world-vectors in σ.

Having decided in Section 3 that the space σ is a six-dimensional (3+3)-spacetime, it follows that world vectors in σ will have six components, and so will be 6-vectors.

Write the 6-vector $\mathscr{A}$ as

$$\mathscr{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$$

and the 6-vector-differential operator analogous to d'Alembert's operator as

$$\bigcirc = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right)$$

where $x_4 = ic\theta_1$, $x_5 = ic\theta_2$, and $x_6 = ic\theta_3$. Define the 6-gradient of the scalar ϕ as

$$\bigcirc\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_4}, \frac{\partial \phi}{\partial x_5}, \frac{\partial \phi}{\partial x_6} \right)$$

the 6-divergence of $\mathscr{A}$ as

$$\bigcirc \cdot \mathscr{A} = \sum_{i=1}^{6} \frac{\partial a_i}{\partial x_i}$$

and the matrix curl of $\mathscr{A}$ as

$$\bigcirc \wedge \mathscr{A} = \begin{pmatrix} 0, b_{12}, b_{13}, \cdots, & b_{16} \\ b_{21}, 0, b_{23}, \cdots, & b_{26} \\ \vdots & \\ b_{61}, \cdots, & b_{65}, 0 \end{pmatrix}$$

where $b_{ij} = \partial a_j/\partial x_i - \partial a_i/\partial x_j$. Of the 36 elements in this matrix, 15 are independent.

It is straightforward to show that if ϕ is differentiable, then

$$\bigcirc \wedge (\bigcirc\phi) \equiv 0$$

and

$$\bigcirc \cdot (\bigcirc \wedge \mathscr{A}) \equiv \bigcirc(\bigcirc \cdot \mathscr{A}) - \bigcirc^2 \mathscr{A}$$

where

$$\bigcirc^2 = \bigcirc \cdot \bigcirc = \sum_{i=1}^{6} \frac{\partial^2}{\partial x_i^2}$$

There are 20 further identities among the b_{ij}; let $\mathbf{a}_{ijk}$ denote the 3-vector whose components along the i, j, and k axes are a_i, a_j, and a_k, respectively. The identities[4] $\nabla \cdot (\nabla \times \mathbf{a}_{ijk}) \equiv 0$ can be written as

$$\frac{\partial b_{jk}}{\partial x_i} + \frac{\partial b_{ki}}{\partial x_j} + \frac{\partial b_{ij}}{\partial x_k} \equiv 0$$

where the (unequal) suffixes run from 1 to 6. Denote these 20 identities by the symbol (ijk). Thus, if the magnetic flux density is introduced as $\mathbf{B} = (B_1, B_2, B_3) = (b_{23}, b_{31}, b_{12})$, the identity (123) gives $\nabla \cdot \mathbf{B} = 0$. This is consistent with setting $a_1 = A_1$, $a_2 = A_2$, $a_3 = A_3$, where A_1, A_2, and A_3 are the three components of the usual vector potential according to our observer in L_2.

Our final assumption is as follows:

A4. The θ variation implied by the operator $\bigcirc$ shall act along the observer's fiber in the direction of increasing t.

From this, if l, m, and n are the direction cosines of the observer's fiber, the components of $\bigcirc$ are $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$, $(l/ic)\,\partial/\partial t$, $(m/ic)\,\partial/\partial t$, $(n/ic)\,\partial/\partial t$.

Again $\mathscr{A}$ and $\mathscr{J}$ are the 6-vector potential and 6-current density, respectively, defined in terms of the observer's measurements. Accordingly,

$$\mathscr{A} = (A_1, A_2, A_3, il\phi/c, im\phi/c, in\phi/c)$$

$$\mathscr{J} = \mu_0(J_1, J_2, J_3, icl\rho, icm\rho, icn\rho)$$

where μ_0 is the permeability of the observer's L-space, ϕ is the scalar potential at a point, ρ is the charge density at the same point, including any apparent charge, and J_1, J_2, J_3 are the components of current density $\mathbf{J}$, all according to our observer.

The remaining calculation becomes more persuasive if the axes in the fiber space are rotated so that $l = 1$, $m = n = 0$, for then $\mathscr{A}$ and $\mathscr{J}$ behave as Rutherford's 4-vectors, and the field equations that correspond to his are

$$\bigcirc \cdot \mathscr{A} = 0, \qquad \bigcirc^2 \mathscr{A} = -\mathscr{J}, \qquad \bigcirc \cdot \mathscr{J} = 0$$

(when no sinks or sources are present).

[4]We have temporarily written $\nabla = (\partial/\partial x_i, \partial/\partial x_j, \partial/\partial x_k)$.

The second equation with one of our identities gives the additional equation

$$\bigcirc \cdot (\bigcirc \wedge \mathscr{A}) = \mathscr{J}$$

It is easily verified that the first equation gives the Lorentz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

where $\mathbf{A} = (A_1, A_2, A_3)$. Again,

$$\bigcirc^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \equiv \Box^2$$

is the d'Alambertian and so the second equation gives

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad \text{and} \qquad \Box^2 \phi = -\rho/\varepsilon_0 \qquad (c^2 = 1/\mu_0 \varepsilon_0)$$

When $\rho = 0$ the last is the wave equation of $(3+1)$-spacetime, so the two principles are not violated in our model. To complete the calculation, we note that the triads of identities [124], [125], [126]; [234], [235], [236]; and [314], [315], [316] give the three components of the Maxwell–Faraday equation

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

and the fourth of the above field equations gives the Maxwell–Ampère equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t \qquad (\mathbf{H} = \mathbf{B}/\mu_0)$$

These results are unaffected by our rotating the axes in fiber space.

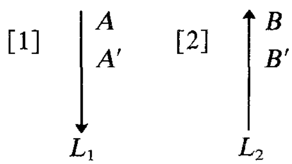
5. DISCUSSION

One of the interesting consequences of the present model is the appearance of the singularities shown in Figure 7b. For example, one might speculate on what would happen if two particles orbiting about one another belong to separate nonparallel L-spaces. To answer this rigorously a dynamics in σ must first be developed, and this has not been done, but if the pair are created at an epoch $t = 0$ earlier than t, in the case when $\beta > 0$, $t_0 < a\beta/\alpha^2$, and initially $r > s_0$ then we must expect the force of attraction to increase indefinitely as the age of the pair approaches the value t_1. This will either result in a collision or more likely will cause an increasing angular acceleration and resulting radiation. One could tentatively suggest that this ageing property lies behind phenomena such as radioactivity.

If in the above case $r < s_0$ initially, no singularity is encountered during ageing and the force just faces gently away.

Another point that should be mentioned is that in line with our view of the role of the observer in the development of the theory, we have assumed that the 3-vectors of the temporal part of the 6-vectors $\mathscr{A}$ and $\mathscr{J}$ are parallel with the observer's fiber. This was because such an assumption is implicit in current physics. However, if an attempt at further generalization is made by allowing the temporal parts of the 6-vectors to have different fixed directions in fiber space, all that happens is that a factor $\cos\xi$ appears in the equations. For example, the charge density ρ in Section 4 becomes $\rho\cos\xi$. Here ξ is the angle between the observer's fiber and the temporal 3-vector of the appropriate 6-vector. It turns out that the two principles still remain unviolated.

A further point is that if $\mathbf{u} = -\mathbf{v}$, then $\beta = -\mathbf{v}\cdot\mathbf{w} = 0$ and $\alpha = 2$. This is the antiparallel case,



If the arrows on the fibers [1] and [2] indicate the positive direction of time, that is, increasing t implies an overall increase in entropy of the two L-spaces L_1 and L_2, and if (A, B) and (A', B') are simultaneous pairs of events, then events in L_1 will appear to an observer in L_2 to have their temporal relationships reversed. In other words, an observer in L_1 will maintain that an event A occurred before A', whereas an observer in L_2 will maintain the reverse. Put more loosely, an observer might think that he had seen a particle "moving backward in time."

There are no doubt many questions that need to be considered, but one outstanding one is whether a body can change from one L-space to another, nonintersecting one. It seems to me that such a change would require a force of some kind. In other words, the problem will arise during the development of a dynamics. Such a possibility could have a profound effect upon our picture of the world, for a change of L-space could imply a change of world-line.

APPENDIX. A NOTE ON SIMULTANEITY IN σ

In Section 3 an extended notion of simultaneity is used, and here we attempt to justify our definition.

While within any given L-space the usual arguments hold for pairs of events, an extension is required when considering two events belonging to different L-spaces. The fibers are assumed to be straight in this preliminary study of the spacetime σ.

In Figure 8 $\overline{0B}$ is a portion of the fiber [2] located at the spatial origin in the frame of the observer in L_2. Suppose that the t scale along this fiber has been chosen with 0 as the origin. Then

$$\overline{0B} = t = |\boldsymbol{\theta}_{\mathrm{B}}|$$

since the coordinates of B in σ are $(\mathbf{0}, \boldsymbol{\theta}_{\mathrm{B}})$.

For the moment let A be any point on the fiber [1], say. We wish to find a *geometrical construction* that will define a scale along $\overline{0A}$ (which may not be a fiber), which is in 1:1 correspondence with the t scale along [2].

Let $\boldsymbol{\theta}_{AB} = \boldsymbol{\theta}_A - \boldsymbol{\theta}_B$ so that

$$\boldsymbol{\theta}_B \cdot \boldsymbol{\theta}_{AB} = \boldsymbol{\theta}_B \cdot \boldsymbol{\theta}_A - t^2$$

We say that if A and B are at the same location, then they are simultaneous if $|\boldsymbol{\theta}_A| = |\boldsymbol{\theta}_B|$.

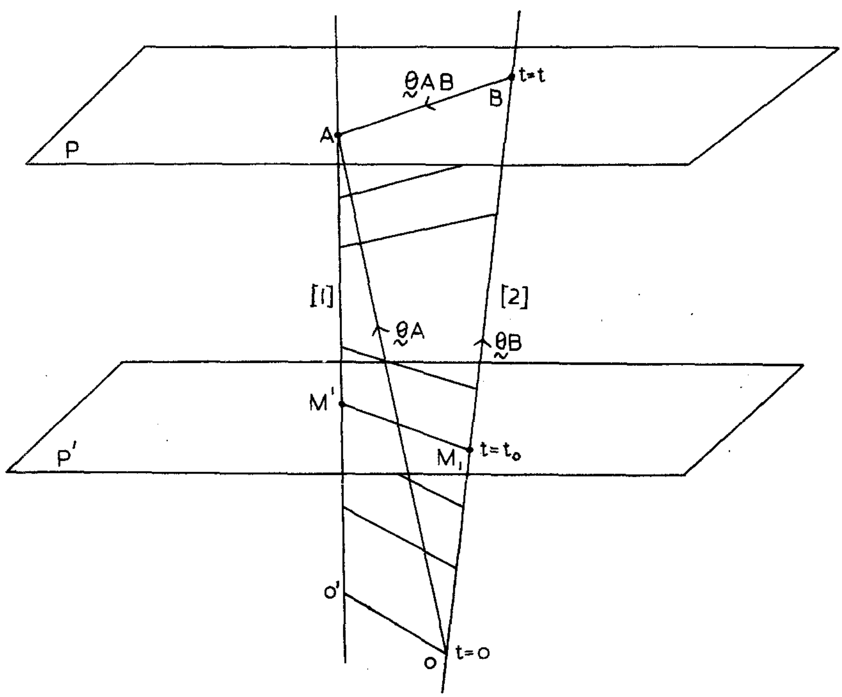


Fig. 8. The simultaneous events A and B belonging to different L-spaces.

It follows at once that when A and B are simultaneous

$$\boldsymbol{\theta}_B \cdot \boldsymbol{\theta}_{AB} = 0$$

that is, $\boldsymbol{\theta}_B$ and $\boldsymbol{\theta}_{AB}$ are orthogonal.

The construction required is made easy if we anticipate the result of Section 3.2 and set $k = 3$, for then the vector $\boldsymbol{\theta}_{AB}$ is seen to lie in the plane P of Figure 8 with [2] as normal. Therefore, if a family of planes, all parallel with P, is arranged to cut [2] at given scale points $\overline{0A}$, any fiber such as [1] will be intersected by this family at corresponding scale points, since it understood that all points of any given plane of the family are mutually simultaneous.

There is no reason to suppose that the fiber [1] passing through A should intersect [2], although it might. There will, however, be a member of the family, P', that cuts [1] and [2] at simultaneous points M' and M, respectively, such that

$$\overline{MM'} = a = \min|\boldsymbol{\theta}_M - \boldsymbol{\theta}_{M'}| \geq 0$$

Having constructed scales along [1] and [2], it is possible to define unit vectors $\mathbf{u}$ and $\mathbf{v}$ along these fibers respectively from $0'$ and 0, the simultaneous pair chosen as time origins.

The event M may be before or after 0. Accordingly, if $\mathbf{v}$ is in the direction of increasing t, and if the time interval between 0 and M is t_0, then $\boldsymbol{\theta}_M = t_0\mathbf{v}$ when M is later than 0, and $\boldsymbol{\theta}_M = -t_0\mathbf{v}$ when M precedes 0.

Unless [1] and [2] are parallel, the scale along [1] will be expanded by comparison with that along [2].

Of course, there is no difficulty, in principle, in extending the definition of simulaneity in fiber space for general k. The family of planes will be replaced by a family of $(k-1)$-flats and the treatment proceeds as above. Accordingly, the argument for $k = 3$ is not circular.

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